

ABOUT DIXMIER'S CONJECTURE

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ABSTRACT. The well-known Dixmier conjecture asks if every algebra endomorphism of the first Weyl algebra over a characteristic zero field is an automorphism.

We bring a hopefully easier to solve conjecture, called the γ, δ conjecture, and show that it is equivalent to the Dixmier conjecture.

Up to checking that in the group generated by automorphisms and anti-automorphisms of A_1 all the involutions belong to one conjugacy class, we show that:

- Every involutive endomorphism from (A_1, γ) to (A_1, δ) is an automorphism (γ and δ are two involutions on A_1).
- Given an endomorphism f of A_1 (not necessarily an involutive endomorphism), if one of $f(X), f(Y)$ is symmetric or skew-symmetric (with respect to any involution on A_1), then f is an automorphism.

1 Introduction

Throughout this paper, K is a characteristic zero field. Let $A_1 = A_1(K) = K\langle X, Y | YX - XY = 1 \rangle$ be the first Weyl algebra over K . Let α be the exchange involution on A_1 given by $\alpha(X) = Y$ and $\alpha(Y) = X$. In [9, Definition 1.1] the notion of an α -endomorphism of A_1 was defined, namely, f is an α -endomorphism of A_1 if f is an algebra endomorphism of A_1 such that $f \circ \alpha = \alpha \circ f$. The α -Dixmier conjecture ($\alpha - D_1$) was posed and proved: Every α -endomorphism of A_1 is an α -automorphism of A_1 , [9, Theorem 2.9]. Actually, it was shown that every α -endomorphism of A_1 is of the following form:

$$f(x) := ax + by + \sum_{j=0}^n S_j \quad \text{and} \quad f(y) := ay + bx + \sum_{j=0}^n S_j.$$

where $n \in \mathbb{N}_0$, $a, b, c_0, \dots, c_n \in K$, with $a^2 - b^2 = 1$, and the S_j 's are defined as follows $S_j := c_j(X - Y)^{2j}$.

Let $K[x, y]$ be the polynomial ring over K and let α be the exchange involution on $K[x, y]$ given by $\alpha(x) = y$ and $\alpha(y) = x$. The α -Jacobian conjecture ($\alpha - JC_2$) was posed and proved: If $f : K[x, y] \rightarrow K[x, y]$ is an α -morphism ($f \circ \alpha = \alpha \circ f$) that satisfies $\text{Jac}(f(x), f(y)) = 1$, then f is invertible, [9, Proposition 4.1].

It is unknown whether $\alpha - JC_2$ implies $\alpha - D_1$, as was mentioned there: "In the previous section we proved the starred Dixmier conjecture (in dimension 1) and in this section we will prove the starred Jacobian conjecture in dimension two, however we do not know if one can deduce the former directly from the latter as in the star-free case".

Remark 1.1. The exchange involution α may be denoted by $X^* = Y$ and $Y^* = X$ instead of by $\alpha(X) = Y$ and $\alpha(Y) = X$; the first notation justifies the name "starred Dixmier conjecture", while the second notation justifies the name " α -Dixmier conjecture".

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2 Equivalence of the Dixmier conjecture and the γ, δ conjecture for A_1 (first version)

Of course, by an α -anti-automorphism of A_1 we mean an anti-automorphism of A_1 that commutes with α . The following proposition will be used in our results.

Proposition 2.1. *Let f be an α -anti-automorphism of A_1 . Then f is an α -anti-automorphism of A_1 . In fact, f is of the following form:*

$$f(x) := ax + by + \sum_{j=0}^n S_j \quad \text{and} \quad f(y) := ay + bx + \sum_{j=0}^n S_j.$$

where $n \in \mathbb{N}_0$, $a, b, c_0, \dots, c_n \in K$, with $a^2 - b^2 = -1$, and the S_j 's are defined as follows $S_j := c_j(X - Y)^{2j}$.

Notice that in the form of an α -endomorphism we have $a^2 - b^2 = 1$, while in the form of an α -anti-automorphism we have $a^2 - b^2 = -1$.

Proof. This result can be obtained in a similar way to the way the result concerning α -endomorphisms was obtained in [9]. \square

In [9] only two involutions were considered: α and β , where β is given by $\beta(X) = X$ and $\beta(Y) = -Y$. Clearly, β is conjugate to α by φ : $\beta := \varphi^{-1} \circ \alpha \circ \varphi$, where $\varphi(X) := \frac{X+Y}{2}$ and $\varphi(Y) := Y - X$. Here we shall consider more involutions:

Definition 2.2 (A γ, δ -endomorphism). Let f be an endomorphism of A_1 and let γ and δ be any two involutions on A_1 . We say that f is

- a γ -endomorphism of A_1 , if $f \circ \gamma = \gamma \circ f$.
- a γ, δ -endomorphism of A_1 , if $f \circ \gamma = \delta \circ f$.

(If $\delta = \gamma$, then a γ, γ -endomorphism is just a γ -endomorphism).

Remarks 2.3. (1) In the above definition we are not assuming that the involutions γ and δ are conjugate to α by an automorphism of A_1 or by an anti-automorphism of A_1 . However, in the first version of the γ, δ conjecture 2.4, we are assuming that each of γ and δ is conjugate to α .

(2) Every automorphism f of A_1 is a γ, δ -endomorphism (with each of γ and δ conjugate to α); just take $\gamma = f^{-1}\alpha f$, $\delta = \alpha$ and get $f\gamma = f(f^{-1}\alpha f) = \alpha f = \delta f$.

(3) If f is a γ, δ -endomorphism, then γ and δ are not necessarily unique such that $f \circ \gamma = \delta \circ f$. For example: If f is an automorphism of A_1 , then the following are legitimate candidates for γ and δ :

$$\begin{aligned} \gamma &= f^{-1}\alpha f, \delta = \alpha. \\ \gamma &= f^{-2}\alpha f^2, \delta = f^{-1}\alpha f. \\ \gamma &= f\alpha f^{-1}, \delta = f^2\alpha f^{-2}. \end{aligned}$$

It is easy to see that there exist infinitely many other options for γ and δ .

Conjecture 2.4 (The γ, δ conjecture (first version)). *For every endomorphism f of A_1 , there exist involutions γ and δ , each of γ and δ is conjugate to α , such that f is a γ, δ -endomorphism of A_1 .*

The following lemma shows why it is good for an endomorphism of A_1 to be a γ, δ -endomorphism, where each of γ and δ is conjugate to α .

Lemma 2.5. *Let f be an endomorphism of A_1 . Then: f is a γ, δ -endomorphism of A_1 , where each of γ and δ is conjugate to $\alpha \iff f$ is an automorphism of A_1 .*

Proof. \Leftarrow : This is just the second remark of Remarks 2.3.

\Rightarrow : f is a γ, δ -endomorphism of A_1 , so $f\gamma = \delta f$.

There exists an automorphism or an anti-automorphism g of A_1 , such that $\gamma = g^{-1}\alpha g$, and there exists an automorphism or an anti-automorphism h of A_1 , such that $\delta = h^{-1}\alpha h$.

Therefore, $f\gamma = \delta f$ becomes $fg^{-1}\alpha g = h^{-1}\alpha hf$, so $(hfg^{-1})\alpha = \alpha(hfg^{-1})$.

- If both g and h are automorphisms or anti-automorphisms, then hfg^{-1} is an α -endomorphism, hence by [9, Theorem 2.9] hfg^{-1} is an α -automorphism. Then obviously f is an automorphism; indeed, $f = (h^{-1}h)f(g^{-1}g) = h^{-1}(hfg^{-1})g$, so f is a composition of three automorphisms: h^{-1} , hfg^{-1} and g , or f is a composition of two anti-automorphisms h^{-1} and g and an automorphism hfg^{-1} .
- If one of g and h is an automorphism and the other is an anti-automorphism, then hfg^{-1} is an α -anti-endomorphism, hence by Proposition 2.1 hfg^{-1} is an α -anti-automorphism. Then obviously f is an automorphism (as a composition of two anti-automorphisms and one automorphism).

□

Of course, our definition of a γ, δ -endomorphism f of A_1 is just another way to say that f is an endomorphism from the ring with involution (A_1, γ) to the ring with involution (A_1, δ) (since, by definition, an endomorphism from the ring with involution (A_1, γ) to the ring with involution (A_1, δ) is an endomorphism of A_1 that satisfies $f \circ \gamma = \delta \circ f$).

So the non-trivial direction of Lemma 2.5 just says the following:

Theorem 2.6 (The involutive Dixmier conjecture is true (first version)). *Let f be an endomorphism from the ring with involution (A_1, γ) to the ring with involution (A_1, δ) , where each of γ and δ is conjugate to α . Then f is an automorphism.*

The computations in the proof of Lemma 2.5 are yielding the following trivial proposition.

Proposition 2.7. *Let f be an endomorphism of A_1 and let γ and δ be two involutions on A_1 , such that $\gamma = g^{-1}\alpha g$ and $\delta = h^{-1}\alpha h$, where both g and h are automorphisms or both are anti-automorphisms or one of g and h is an automorphism and the other is an anti-automorphism.*

Then the following statements are equivalent:

- (1) f is a γ, δ -endomorphism of A_1 .
- (2) hfg^{-1} is an α -endomorphism (when both g and h are automorphisms or both are anti-automorphisms) or an α -anti-endomorphism (when one of g and h is an automorphism and the other is an anti-automorphism).

Proof. Clear.

□

Given an endomorphism f of A_1 , each of the statements of Proposition 2.7 is equivalent to f being an automorphism:

- (1) Lemma 2.5 shows that being a γ, δ -endomorphism of A_1 (with each of γ and δ conjugate to α) is equivalent to being an automorphism of A_1 .
- (2) First option: hfg^{-1} is an α -endomorphism (both g and h are automorphisms or both are anti-automorphisms), then from [9, Theorem 2.9] hfg^{-1} is an α -automorphism, hence f is an automorphism.

Second option: hfg^{-1} is an α -anti-endomorphism (one of g and h is an automorphism and the other is an anti-automorphism), then from Proposition 2.1 hfg^{-1} is an α -anti-automorphism, hence f is an automorphism.

The opposite direction is trivial: If f is an automorphism of A_1 , then taking $g = f$ and $h = 1$ yields $hfg^{-1} = 1ff^{-1} = 1$ which is an α -endomorphism.

In other words, given an endomorphism f of A_1 , in order to get a positive answer to the Dixmier conjecture, it is sufficient to show one of the following:

- (1) f is a γ, δ -endomorphism of A_1 (with each of γ and δ conjugate to α).
- (2) There exist g and h both are automorphisms or both are anti-automorphisms of A_1 such that hfg^{-1} is an α -endomorphism of A_1 ;
or there exist g and h , one of g and h is an automorphism of A_1 and the other is an anti-automorphism of A_1 such that hfg^{-1} an α -anti-endomorphism of A_1 .

Summarizing, the connection between the Dixmier conjecture and the γ, δ conjecture (first version) is as follows:

Theorem 2.8 (Equivalence of the Dixmier conjecture and the γ, δ conjecture (first version)). *The Dixmier conjecture is true \iff The γ, δ conjecture (first version) is true.*

Proof. \implies : Let f be an endomorphism of A_1 . We must find involutions γ and δ , each of γ and δ is conjugate to α , such that $f \circ \gamma = \delta \circ f$. The Dixmier conjecture is true, hence f is an automorphism, so use Lemma 2.5.

\impliedby : Let f be an endomorphism of A_1 . We must show that f is an automorphism of A_1 . The γ, δ conjecture (first version) is true, hence there exist involutions γ and δ , each of γ and δ is conjugate to α , such that f is a γ, δ -endomorphism of A_1 . Now use Lemma 2.5. \square

And similarly,

Conjecture 2.9 (The g, h conjecture (first version)). *For every endomorphism f of A_1 , there exist g and h , both automorphisms, both anti-automorphisms or one is an automorphism and the other is an anti-automorphism, such that hfg^{-1} is an α -endomorphism (when both g and h are automorphisms or both are anti-automorphisms) or an α -anti-endomorphism (when one of g and h is an automorphism and the other is an anti-automorphism).*

Theorem 2.10 (Equivalence of the Dixmier conjecture and the g, h conjecture (first version)). *The Dixmier conjecture is true \iff The g, h conjecture (first version) is true.*

2.1 An extension condition and a restriction condition (first version)

In view of the above results, our (hopefully possible) mission is to prove that the γ, δ conjecture (first version) is true or that the g, h conjecture (first version) is true; Thus far, given an endomorphism f of A_1 , we only managed to find two conditions, an extension condition and a restriction condition, each implies that f is a γ, δ -endomorphism of A_1 (each of γ and δ is conjugate to α), hence an automorphism of A_1 , see Theorem 2.12 and Theorem 2.14.

We have not yet managed to find a condition that implies directly that the g, h conjecture (first version) is true: We only suggest that instead of considering a general endomorphism f of A_1 , one will consider an irreducible endomorphism f of A_1 , see [7, Definition 3.1] for the definition of an irreducible endomorphism.

Namely, we suggest that for every irreducible endomorphism f of A_1 , one will find g and h such that hfg^{-1} is an α -endomorphism or an α -anti-endomorphism of A_1 . Hence, hfg^{-1} is onto, so f is an automorphism. Therefore, we have shown

that every irreducible endomorphism of A_1 is an automorphism of A_1 , concluding that the Dixmier conjecture is true, by changing a little [7, Theorem 3.3].

Now, let f be an endomorphism of A_1 . From now on we will use the following notations: $P := f(X)$ and $Q := f(Y)$. So, $1 = [f(Y), f(X)] = [Q, P]$. The image of f , $K\langle P, Q | QP - PQ = 1 \rangle$, is a subalgebra of A_1 which is isomorphic to A_1 ; we will denote the image of f by T .

We can define an involution γ on T by $\gamma(P) = Q$, $\gamma(Q) = P$ (and extended in the obvious way to all of T).

U. Vishne raised the following question, as a general question about involutions on A_1 , without intending to connect it to a solution to the Dixmier conjecture: “Given an endomorphism f of A_1 , with $P := f(X)$ and $Q := f(Y)$, can the involution γ ($\gamma(P) = Q$, $\gamma(Q) = P$) on T be extended to A_1 ?”.

(If the given endomorphism f of A_1 is already an automorphism of A_1 , then γ is already an involution on A_1).

We suggest to adapt Vishne's question in a way that connects it to a solution to the Dixmier conjecture. This is done in Theorem 2.12.

Definition 2.11 (The extension condition (first version)). Let f be an endomorphism of A_1 . We say that f satisfies the extension condition (first version) if the involution γ ($\gamma(P) = Q$, $\gamma(Q) = P$) on T can be extended to A_1 , where the extended involution is conjugate to α .

Theorem 2.12. *Let f be an endomorphism of A_1 . If f satisfies the extension condition (first version), then f is an automorphism of A_1 .*

Proof. f satisfies the extension condition (first version), so there exists an involution on A_1 , call it Γ , such that Γ extends γ , and Γ is conjugate to α .

Therefore, we get that f is an α, Γ -endomorphism of A_1 (Γ is conjugate to α), because: $(f\alpha)(X) = f(Y) = Q = \gamma(P) = \Gamma(P) = \Gamma(f(X)) = (\Gamma f)(X)$ and $(f\alpha)(Y) = f(X) = P = \gamma(Q) = \Gamma(Q) = \Gamma(f(Y)) = (\Gamma f)(Y)$.

Hence, from Lemma 2.5 we get that f is an automorphism of A_1 . □

We also suggest to consider the following “restriction condition”.

Definition 2.13 (The restriction condition (first version)). Let f be an endomorphism of A_1 . We say that f satisfies the restriction condition (first version) if the exchange involution α on A_1 , when restricted to T , is an involution on T , and γ is conjugate by an automorphism or by an anti-automorphism of T to the restriction of α to T .

Notice that, in general, the restriction of α to an affine subalgebra of A_1 need not be an involution on the affine subalgebra, since α may take the generators of the affine subalgebra outside it.

Theorem 2.14. *Let f be an endomorphism of A_1 . If f satisfies the restriction condition (first version), then f is an automorphism of A_1 .*

Proof. f satisfies the restriction condition (first version), so α restricted to T is an involution on T , and γ is conjugate by an automorphism or by an anti-automorphism of T to (the restriction of) α :

$\gamma = g^{-1}\alpha g$, where g is an automorphism or an anti-automorphism of T (α is the restriction of α to T).

Therefore, we get that f is an α, γ -endomorphism of A_1 (γ is conjugate to α), because: $(f\alpha)(X) = f(Y) = Q = \gamma(P) = \gamma(f(X)) = (\gamma f)(X)$ and $(f\alpha)(Y) = f(X) = P = \gamma(Q) = \gamma(f(Y)) = (\gamma f)(Y)$.

Hence, from Lemma 2.5 we get that f is an automorphism of A_1 . □

Remark 2.15. In the proof of Theorem 2.14, γ is conjugate by g to the restriction of α to T , where g is an automorphism or an anti-automorphism of T , while in our definition and results about γ, δ -endomorphisms we demanded that each of γ and δ is conjugate by an automorphism or by an anti-automorphism of A_1 , to α .

However, one can see that if f is an α, δ -endomorphism, where $\delta = h^{-1}\alpha h$, then it is enough to demand that h is an automorphism or an anti-automorphism of the image of f , instead of demanding that h is an automorphism or an anti-automorphism of A_1 .

Now, instead of defining the involution γ on T ($\gamma(P) = Q$, $\gamma(Q) = P$), we could have defined on T , for example, $\epsilon(P) = P$, $\epsilon(Q) = -Q$, extend in the obvious way to all of T , and get an involution on T .

Let E be the involution on A_1 given by $E(X) = X$, $E(Y) = -Y$ (E is just the involution β we have already mentioned). It is easy to see that $fE = \epsilon f$:

$$\begin{aligned} (fE)(X) &= f(E(X)) = f(X) = P = \epsilon(P) = \epsilon(f(X)) = (\epsilon f)(X) \text{ and} \\ (fE)(Y) &= f(E(Y)) = f(-Y) = -Q = \epsilon(Q) = \epsilon(f(Y)) = (\epsilon f)(Y). \end{aligned}$$

Then taking E and ϵ instead of α and γ , yield similar extension results, namely, if the involution ϵ on T can be extended to A_1 , where the extended involution is conjugate to E , then f is an automorphism of A_1 ; indeed, $fE = \epsilon f$. Let $\tilde{\epsilon}$ be the extension of ϵ to A_1 , and let $\tilde{\epsilon} = g^{-1}Eg$, where g is an automorphism or an anti-automorphism of A_1 . Clearly, $fE = \tilde{\epsilon}f$. We have already seen that E is conjugate to α by φ : $E = \varphi^{-1}\alpha\varphi$ ($\varphi(X) := \frac{X+Y}{2}$ and $\varphi(Y) := Y - X$). Therefore, $fE = \tilde{\epsilon}f$ becomes $f\varphi^{-1}\alpha\varphi = g^{-1}Egf$, so $f\varphi^{-1}\alpha\varphi = g^{-1}(\varphi^{-1}\alpha\varphi)gf$. Then, $(\varphi gf\varphi^{-1})\alpha = \alpha(\varphi gf\varphi^{-1})$.

- If g is an automorphism of A_1 , then $\varphi gf\varphi^{-1}$ is an α -endomorphism of A_1 , hence by [9, Theorem 2.9] an automorphism of A_1 . Then clearly f is an automorphism of A_1 .
- If g is an anti-automorphism of A_1 , then $\varphi gf\varphi^{-1}$ is an α -anti-endomorphism of A_1 , hence by Proposition 2.1 an anti-automorphism of A_1 . Then clearly f is an automorphism of A_1 .

More generally, take any involution E on A_1 that is conjugate to α . Write $E(X) = \sum \lambda_{ij}X^iY^j$ and $E(Y) = \sum \mu_{ij}X^iY^j$. Then it is clear that $\epsilon(P) := \sum \lambda_{ij}P^iQ^j$ and $\epsilon(Q) := \sum \mu_{ij}P^iQ^j$ is an involution on T . It is easy to see that $fE = \epsilon f$:

$$\begin{aligned} (fE)(X) &= f(\sum \lambda_{ij}X^iY^j) = \sum \lambda_{ij}f(X)^if(Y)^j = \\ &= \sum \lambda_{ij}P^iQ^j = \epsilon(P) = \epsilon(f(X)) = (\epsilon f)(X) \text{ and} \\ (fE)(Y) &= f(\sum \mu_{ij}X^iY^j) = \sum \mu_{ij}f(X)^if(Y)^j = \\ &= \sum \mu_{ij}P^iQ^j = \epsilon(Q) = \epsilon(f(Y)) = (\epsilon f)(Y). \end{aligned}$$

Again, if the involution ϵ on T can be extended to A_1 , where the extended involution is conjugate to E , then f is an automorphism of A_1 (just make the appropriate changes in the proof of the above example: $E(X) = X$, $E(Y) = -Y$, $\epsilon(P) = P$, $\epsilon(Q) = -Q$).

Therefore, we have:

Definition 2.16 (The generalized extension condition (first version)). Let f be an endomorphism of A_1 . We say that f satisfies the generalized extension condition (first version), if there exists an involution E on A_1 that is conjugate to α (write $E(X) = \sum \lambda_{ij}X^iY^j$ and $E(Y) = \sum \mu_{ij}X^iY^j$) such that its corresponding involution ϵ on T ($\epsilon(P) := \sum \lambda_{ij}P^iQ^j$ and $\epsilon(Q) := \sum \mu_{ij}P^iQ^j$) can be extended to A_1 , where the extended involution is conjugate to E .

Theorem 2.17. Let f be an endomorphism of A_1 . If f satisfies the generalized extension condition (first version), then f is an automorphism of A_1 .

Similarly, instead of demanding that α restricted to T is an involution on T and γ is conjugate by an automorphism or by an anti-automorphism of T to the restriction of α to T , demand that E restricted to T is an involution on T and ϵ is conjugate by an automorphism or by an anti-automorphism of T to the restriction of E to T .

Then taking E and ϵ instead of α and γ , yield similar restriction results:

Definition 2.18 (The generalized restriction condition (first version)). Let f be an endomorphism of A_1 . We say that f satisfies the generalized restriction condition (first version), if there exists an involution E on A_1 that is conjugate to α , such that E restricted to T is an involution on T .

Theorem 2.19. *Let f be an endomorphism of A_1 . If f satisfies the generalized restriction condition (first version), then f is an automorphism of A_1 .*

For a detailed proof, see the proof of Theorem 3.12 and adjust it to the first version.

3 Equivalence of the Dixmier conjecture and the γ, δ conjecture for A_1 (second version)

The reader has probably wondered why, in the previous section, “first version” was added to most of our results. The reason is that we wish to bring those results without the conjugacy conditions, where the conjugacy conditions are: “each of γ and δ is conjugate to α ”, “where the extended involution is conjugate to α ” and “ γ is conjugate by an automorphism or by an anti-automorphism of T to (the restriction of) α ”.

The “second version” results are just the first version results without the conjugacy conditions.

We now explain how to remove the conjugacy conditions, without losing our previous, first version, results:

An interesting problem is finding all the involutions on A_1 ; Of course, conjugates of α by an automorphism or by an anti-automorphism are involutions. Apriori, it may happen that there exist involutions other than conjugates of α . However, in a private note [3], J. Bell has sketched an idea of proof that there are no involutions on A_1 other than the conjugates of α . His idea includes two steps; the first step has not yet been checked, though it seems to be true. The second step relies on the first step.

- First step: Recall that the group of automorphisms of A_1 is the amalgamated free product of two of its subgroups (denote these two subgroups by A and B , where A is the linear automorphisms and B is the triangular automorphisms) over their intersection.

Similarly, it seems that the group G generated by automorphisms and anti-automorphisms of A_1 is the amalgamated free product of two of its subgroups (denote these two subgroups by \tilde{A} and \tilde{B} , each is defined similarly to the above A and B) over their intersection. (Notice that the group of automorphisms of A_1 is a normal subgroup of G , having index 2).

- Second step: Applying Bass-Serre theorem [10, Corollary 1, page 6] to a given involution γ on A_1 shows almost immediately that it is conjugate to α ; indeed, γ is of order 2, so it is conjugate to an order 2 element $a \in \tilde{A}$ or it is conjugate to an order 2 element $b \in \tilde{B}$. Since there are no elements of order 2 in \tilde{B} , γ is conjugate to an order 2 element $a \in \tilde{A}$. Finally, any order 2 element $a \in \tilde{A}$ is conjugate to α .

If indeed there are no involutions on A_1 other than the conjugates of α , then in our first version results we can omit the conjugacy conditions and get the following results:

Conjecture 3.1 (The γ, δ conjecture (second version)). *For every endomorphism f of A_1 , there exist involutions γ and δ , such that f is a γ, δ -endomorphism of A_1 .*

Lemma 3.2. *Let f be an endomorphism of A_1 . Then: f is a γ, δ -endomorphism of $A_1 \iff f$ is an automorphism of A_1 .*

The following theorem shows that the Dixmier conjecture is true in the category of rings with involution:

Theorem 3.3 (The involutive Dixmier conjecture is true (second version)). *Let f be an endomorphism from the ring with involution (A_1, γ) to the ring with involution (A_1, δ) . Then f is an automorphism.*

Theorem 3.4 (Equivalence of the Dixmier conjecture and the γ, δ conjecture (second version)). *The Dixmier conjecture is true \iff The γ, δ conjecture (second version) is true.*

Definition 3.5 (The extension condition (second version)). Let f be an endomorphism of A_1 . We say that f satisfies the extension condition (second version) if the involution γ on T ($\gamma(P) = Q, \gamma(Q) = P$) can be extended to A_1 .

Theorem 3.6. *Let f be an endomorphism of A_1 . If f satisfies the extension condition (second version), then f is an automorphism of A_1 .*

Definition 3.7 (The generalized extension condition (second version)). Let f be an endomorphism of A_1 . We say that f satisfies the generalized extension condition (second version), if there exists an involution E on A_1 (write $E(X) = \sum \lambda_{ij} X^i Y^j$ and $E(Y) = \sum \mu_{ij} X^i Y^j$) such that its corresponding involution ϵ on T ($\epsilon(P) := \sum \lambda_{ij} P^i Q^j$ and $\epsilon(Q) := \sum \mu_{ij} P^i Q^j$) can be extended to A_1 .

Theorem 3.8. *Let f be an endomorphism of A_1 . If f satisfies the generalized extension condition (second version), then f is an automorphism of A_1 .*

We move to the restriction condition results; in contrast to the extension condition results, here we have found a condition on f which implies that f satisfies the restriction condition (hence f is an automorphism), namely that one of $f(X), f(Y)$ is symmetric or skew-symmetric with respect to any involution on A_1 , see Theorem 3.14. However, we bring a second proof for Theorem 3.14, which is independent of the generalized restriction condition.

Definition 3.9 (The restriction condition (second version)). Let f be an endomorphism of A_1 . We say that f satisfies the restriction condition (second version) if the exchange involution α on A_1 , when restricted to T , is an involution on T .

Theorem 3.10. *Let f be an endomorphism of A_1 . If f satisfies the restriction condition (second version), then f is an automorphism of A_1 .*

Proof. f satisfies the restriction condition (second version), so α restricted to T is an involution on T . Since T is isomorphic to A_1 and we “assume” that Bell’s first step is indeed true, it follows that every involution on T is conjugate to one chosen involution on T .

Let the restriction of α to T be the “chosen” involution on T . Since γ is an involution on T , we get that γ is conjugate to the restriction of α to T : $\gamma = g^{-1} \alpha g$, where g is an automorphism or an anti-automorphism of T (α is the restriction of α to T).

Therefore, we get that f is an α, γ -endomorphism of A_1 (γ is conjugate to α), because: $(f\alpha)(X) = f(Y) = Q = \gamma(P) = \gamma(f(X)) = (\gamma f)(X)$ and $(f\alpha)(Y) = f(X) = P = \gamma(Q) = \gamma(f(Y)) = (\gamma f)(Y)$.

Hence, from Lemma 2.5 we get that f is an automorphism of A_1 . \square

Definition 3.11 (The generalized restriction condition (second version)). Let f be an endomorphism of A_1 . We say that f satisfies the generalized restriction condition (second version), if there exists an involution E on A_1 , such that E restricted to T is an involution on T .

Theorem 3.12. *Let f be an endomorphism of A_1 . If f satisfies the generalized restriction condition (second version), then f is an automorphism of A_1 .*

Proof. f satisfies the generalized restriction condition (second version), so there exists an involution E on A_1 such that E restricted to T is an involution on T . Since T is isomorphic to A_1 and we “assume” that Bell’s first step is indeed true, it follows that every involution on T is conjugate to one chosen involution on T .

Let the restriction of E to T be the “chosen” involution on T .

Write $E(X) = \sum \lambda_{ij} X^i Y^j$ and $E(Y) = \sum \mu_{ij} X^i Y^j$. Its corresponding involution ϵ on T is defined to be $\epsilon(P) := \sum \lambda_{ij} P^i Q^j$ and $\epsilon(Q) := \sum \mu_{ij} P^i Q^j$.

Since ϵ is an involution on T , we get that ϵ is conjugate to the restriction of E to T : $\epsilon = g^{-1} E g$, where g is an automorphism or an anti-automorphism of T (E is the restriction of E to T).

Therefore, we get that f is an E, ϵ -endomorphism of A_1 because:

$$\begin{aligned} (fE)(X) &= f(\sum \lambda_{ij} X^i Y^j) = \sum \lambda_{ij} f(X)^i f(Y)^j = \\ &= \sum \lambda_{ij} P^i Q^j = \epsilon(P) = \epsilon(f(X)) = (\epsilon f)(X) \text{ and} \\ (fE)(Y) &= f(\sum \mu_{ij} X^i Y^j) = \sum \mu_{ij} f(X)^i f(Y)^j = \\ &= \sum \mu_{ij} P^i Q^j = \epsilon(Q) = \epsilon(f(Y)) = (\epsilon f)(Y). \end{aligned}$$

Now E is conjugate to α by our “assumption” in this section that there are no involutions on A_1 other than the conjugates of α , and $\epsilon = g^{-1} E g$. Write $E = h^{-1} \alpha h$, where h is an automorphism or an anti-automorphism of A_1 . Then $fE = \epsilon f$ becomes $f h^{-1} \alpha h = g^{-1} E g f = g^{-1} (h^{-1} \alpha h) g f = g^{-1} h^{-1} \alpha h g f$, so $f h^{-1} \alpha h = g^{-1} h^{-1} \alpha h g f$. Then $h g f h^{-1} \alpha = \alpha h g f h^{-1}$:

- If g is an automorphism of T , then $h g f h^{-1}$ is an α -endomorphism of A_1 , so [9, Theorem 2.9] implies that $h g f h^{-1}$ is an (α) -automorphism of A_1 . Therefore, f is an automorphism of A_1 .
- If g is an anti-automorphism of T , then $h g f h^{-1}$ is an α -anti-endomorphism of A_1 , so Proposition 2.1 implies that $h g f h^{-1}$ is an (α) -anti-automorphism of A_1 .

Therefore, f is an automorphism of A_1 . \square

Let ϵ be an involution on A_1 and let $w \in A_1$. w is symmetric if $\epsilon(w) = w$, and w is skew-symmetric if $\epsilon(w) = -w$.

Now, in case one of $f(X), f(Y)$ is symmetric or skew-symmetric with respect to α , the restriction condition is satisfied by f :

Theorem 3.13. *Let f be an endomorphism of A_1 . Assume that one of the following conditions is satisfied:*

- P is symmetric.
- P is skew-symmetric.
- Q is symmetric.
- Q is skew-symmetric.

Where by symmetric or skew-symmetric we mean symmetric or skew-symmetric with respect to α . Then f is an automorphism of A_1 .

The commutator of two symmetric or two skew-symmetric elements is skew-symmetric, hence it is impossible to have both P and Q symmetric or both P and Q skew-symmetric (since $[Q, P] = 1$ and 1 is not skew-symmetric). But it may happen that P is symmetric and Q is skew-symmetric or vice-versa, for example: $P = \frac{X+Y}{2}$ and $Q = Y - X$.

Actually, if P is symmetric and Q is skew-symmetric (or vice-versa), then it is immediate that f is an automorphism; Write $P = S$ and $Q = K$ with S symmetric and K skew-symmetric. Define $g(X) = (1/\sqrt{2})(S - K)$ and $g(Y) = (1/\sqrt{2})(S + K)$.

g is an endomorphism of A_1 :

$$[g(Y), g(X)] = [(1/\sqrt{2})(S + K), (1/\sqrt{2})(S - K)] = (1/2)[S + K, S - K] = (1/2)[S, -K] + (1/2)[K, S] = (1/2)[K, S] + (1/2)[K, S] = [K, S] = [Q, P] = 1.$$

g is an α -endomorphism of A_1 :

$$(g\alpha)(X) = g(Y) = (1/\sqrt{2})(S + K) = (1/\sqrt{2})\alpha(S - K) = \alpha((1/\sqrt{2})(S - K)) = \alpha(g(X)) = (\alpha g)(X),$$

$$(g\alpha)(Y) = g(X) = (1/\sqrt{2})(S - K) = (1/\sqrt{2})\alpha(S + K) = \alpha((1/\sqrt{2})(S + K)) = \alpha(g(Y)) = (\alpha g)(Y).$$

From [9, Theorem 2.9] g is an automorphism of A_1 , hence it is clear that f is an automorphism of A_1 .

Proof. Assume that P is symmetric, namely $\alpha(P) = P$. From $[Q, P] = 1$ we get $[P, \alpha(Q)] = [\alpha(P), \alpha(Q)] = \alpha([Q, P]) = \alpha(1) = 1$. Then $[P, -Q - \alpha(Q)] = [P, -Q] - [P, \alpha(Q)] = 1 - 1 = 0$, so $-Q - \alpha(Q)$ is in the centralizer of P . The centralizer of P is the polynomial algebra $K[P]$, see [6], hence there exists a polynomial $h(t) \in K[t]$ such that $-Q - \alpha(Q) = h(P)$, so $\alpha(Q) = -Q - h(P)$. This implies that α restricted to T is an involution on T ; indeed, $\alpha(P) = P \in T$ and $\alpha(Q) = -Q - h(P) \in T$, so α restricted to T is an endomorphism of T . Then it is clear that α restricted to T is an involution on T ; just for fun:

α restricted to T is of order 2: $\alpha(\alpha(Q)) = \alpha(-Q - h(P)) = \alpha(-Q) + \alpha(-h(P)) = -\alpha(Q) - \alpha(h(P)) = -(-Q - h(P)) - h(P) = Q + h(P) - h(P) = Q$ (of course $\alpha(h(P)) = h(P)$, since $\alpha(P) = P$).

Therefore, f satisfies the restriction condition (second version), so from Theorem 3.10 f is an automorphism of A_1 .

Showing that each of the other three conditions implies that f is an automorphism of A_1 , is similar. \square

Theorem 3.14. *Let f be an endomorphism of A_1 . Assume that one of the following conditions is satisfied:*

- P is symmetric.
- P is skew-symmetric.
- Q is symmetric.
- Q is skew-symmetric.

Where by symmetric or skew-symmetric we mean symmetric or skew-symmetric with respect to any involution on A_1 . Then f is an automorphism of A_1 .

Proof. Assume that P is symmetric with respect to $E = g^{-1}\alpha g$ (g is an automorphism or an anti-automorphism of A_1), namely $E(P) = P$. From $[Q, P] = 1$ we get $[P, E(Q)] = [E(P), E(Q)] = E([Q, P]) = E(1) = 1$. Then $[P, -Q - E(Q)] = [P, -Q] - [P, E(Q)] = 1 - 1 = 0$, so $-Q - E(Q)$ is in the centralizer of P . The centralizer of P is the polynomial algebra $K[P]$, see [6], hence there exists a polynomial $h(t) \in K[t]$ such that $-Q - E(Q) = h(P)$, so $E(Q) = -Q - h(P)$. This implies that E restricted to T is an involution on T ; indeed, $E(P) = P \in T$ and $E(Q) = -Q - h(P) \in T$, so E restricted to T is an endomorphism of T . Then it is clear that E restricted to T is an involution on T ; just for fun:

E restricted to T is of order 2: $E(E(Q)) = E(-Q - h(P)) = E(-Q) + E(-h(P)) = -E(Q) - E(h(P)) = -(-Q - h(P)) - h(P) = Q + h(P) - h(P) = Q$ (of course $E(h(P)) = h(P)$, since $E(P) = P$).

Therefore, f satisfies the generalized restriction condition (second version), so from Theorem 3.12 f is an automorphism of A_1 .

Showing that each of the other three conditions implies that f is an automorphism of A_1 , is similar.

Second proof, independent of the generalized restriction condition:

Assume that P is symmetric with respect to $E = g^{-1}\alpha g$ (g is an automorphism or an anti-automorphism of A_1), namely $E(P) = P$.

Write: $P = S_1$ and $Q = S_2 + K_2$, where S_1 and S_2 are symmetric and K_2 is skew-symmetric, with respect to E .

From $1 = [Q, P]$ we get $1 = [S_2 + K_2, S_1] = [S_2, S_1] + [K_2, S_1]$. $[S_2, S_1]$ is skew-symmetric and $[K_2, S_1]$ is symmetric, hence $[S_2, S_1] = 0$ and $[K_2, S_1] = 1$.

$[S_2, S_1] = 0$, so S_2 is in the centralizer of S_1 . The centralizer of S_1 is the polynomial algebra $K[S_1]$, see [6], hence there exists a polynomial $h(t) \in K[t]$ such that $S_2 = h(S_1)$.

$[K_2, S_1] = 1$, so $g(X) := (1/\sqrt{2})(S_1 - K_2)$ and $g(Y) := (1/\sqrt{2})(S_1 + K_2)$ defines an E -endomorphism of A_1 , which is an automorphism of A_1 (by Lemma 2.5). Then obviously S_1 and K_2 generate all A_1 .

Of course, $K_2 = S_2 + K_2 - S_2 = Q - h(S_1) = Q - h(P)$ is in T . So S_1 and K_2 are in T , which implies that $T = A_1$, hence f is an automorphism of A_1 .

Showing that each of the other three conditions implies that f is an automorphism of A_1 , is similar.

□

The first condition in Theorem 3.14 is equivalent to the condition that there exists an automorphism or an anti-automorphism g of A_1 such that $g(P)$ is symmetric with respect to α (and similarly the other three conditions).

Proposition 3.15. *Let f be an endomorphism of A_1 . The following conditions are equivalent:*

- (1) *There exists an involution E on A_1 such that P is symmetric (or skew-symmetric) with respect to E .*
- (2) *There exists an automorphism or an anti-automorphism g of A_1 such that $g(P)$ is symmetric (or skew-symmetric) with respect to α .*

Of course, the analogue result when taking Q instead of P , is also true.

Proof. (1) \Rightarrow (2): Let E be an involution on A_1 such that P is symmetric with respect to E , namely $E(P) = P$. By our “assumption” E is conjugate to α , so there exists an automorphism or an anti-automorphism g of A_1 such that $E = g^{-1}\alpha g$. Hence, $E(P) = P$ becomes $(g^{-1}\alpha g)(P) = P$, so $\alpha(g(P)) = g(P)$, which means that $g(P)$ is symmetric with respect to α .

The skew-symmetric case is similar.

(2) \Rightarrow (1): Let g be an automorphism or an anti-automorphism of A_1 such that $g(P)$ is symmetric with respect to α , namely $\alpha(g(P)) = g(P)$. Then $(g^{-1}\alpha g)(P) = P$, which means that P is symmetric with respect to the involution $g^{-1}\alpha g$ on A_1 .

The skew-symmetric case is similar.

□

From Proposition 3.15, given an endomorphism f of A_1 , one wishes to find linear and triangular automorphisms and linear and triangular anti-automorphisms g_1, \dots, g_n such that $(g_1 \cdots g_n)(P)$ is symmetric or skew-symmetric with respect to α . (Or one wishes to find linear and triangular automorphisms and linear and

triangular anti-automorphisms g_1, \dots, g_n such that $(g_1 \cdots g_n)(Q)$ is symmetric or skew-symmetric with respect to α .

Next, the second version of the g, h conjecture is exactly the same as the first version of the g, h conjecture:

Conjecture 3.16 (The g, h conjecture (second version)). *For every endomorphism f of A_1 , there exist g and h , both automorphisms, both anti-automorphisms or one is an automorphism and the other is an anti-automorphism, such that hfg^{-1} is an α -endomorphism (when both g and h are automorphisms or both are anti-automorphisms) or an α -anti-endomorphism (when one of g and h is an automorphism and the other is an anti-automorphism).*

Theorem 3.17 (Equivalence of the Dixmier conjecture and the g, h conjecture (second version)). *The Dixmier conjecture is true \iff The g, h conjecture (second version) is true.*

4 Analogous results for $K[x, y]$

The above Bell's idea concerning involutions on A_1 can be applied to $K[x, y]$.

Apriori, we define an involution γ on $k[x, y]$ as an automorphism of order 2, without demanding that the Jacobian of $\gamma(x), \gamma(y)$ will equal -1 (of course, the Jacobian of $\gamma(x), \gamma(y)$ is a non-zero scalar, since γ is invertible). However, Bell's idea shows that γ is conjugate to α , hence we get for free that the Jacobian of $\gamma(x), \gamma(y)$ equals -1 (since the Jacobian of $\alpha(x), \alpha(y)$ equals -1).

Proposition 4.1. *There are no involutions on $K[x, y]$ other than the conjugates of α .*

Proof. Since $K[x, y]$ is commutative, anti-automorphisms are automorphisms, so every involution on $K[x, y]$ is an order 2 automorphism.

It is well-known that the group of automorphisms of $K[x, y]$ is the amalgamated free product of A and B over their intersection, where A is the affine automorphisms and B is the triangular automorphisms; this result appeared in many articles whose authors are: van der Kulk, Kambayashi, Nagata, Dicks, Alperin and Shafarevich (the list of articles can be found in [12, page 114]).

Therefore, one can immediately apply Bass-Serre theorem [10, Corollary 1, page 6] to a given involution γ on $K[x, y]$ and get that γ is conjugate to α ; indeed, γ is of order 2, so it is conjugate to an order 2 element $a \in A$ or it is conjugate to an order 2 element $b \in B$. Since there are no elements of order 2 in B , γ is conjugate to an order 2 element $a \in A$. Finally, any order 2 element $a \in A$ is conjugate to α . \square

We can obtain for $K[x, y]$ similar results to the results of A_1 . We leave to the reader the task of checking this.

We just mention that it seems that now we have two advantages in comparison to A_1 :

- By Proposition 4.1, we do know that there are no involutions on A_1 other than the conjugates of α , so we can obtain the results for $K[x, y]$ immediately from the second version results of A_1 .

In particular, every morphism f from the ring with involution $(K[x, y], \gamma)$ to the ring with involution $(K[x, y], \delta)$ that satisfies $\text{Jac}(f(x), f(y)) \in K^*$ is invertible.

- The problem of extending an involution from $K[p, q]$ to $K[x, y]$ (given a morphism f of $K[x, y]$ with $p = f(x)$ and $q = f(y)$) seems a little less difficult problem, since now we deal with commutative rings. However, the

condition on the Jacobian may add some new difficulty, since we wish that the Jacobian of the extension will equal -1 .

The Jacobian conjecture for $K[x, y]$ implies the Dixmier conjecture for A_1 . This was proved by Tsuchimoto [11], Belov and Kontsevich [4] and Bavula [2] (actually, the generalized version was proved in those papers).

Therefore, if one can prove that the γ, δ conjecture for $K[x, y]$ is true (hence by our results the Jacobian conjecture for $K[x, y]$ is true), then the dixmier conjecture for A_1 is true.

Finally, we suggest to check if all the results brought in this article are applicable to Poisson algebras. If so, in view of [1], it will be interesting to see if the γ, δ conjecture is true there.

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